Joe Dinius MATH 527A
HW8 December 2, 2008

1. **Local base, base and subbase**

(a) Exercise 3.4.7. (This is the statement that we can define topologies through bases as described in class)

Prove Proposition 3.4.2, following the argument in (3.1.1). The collection $\mathcal{B}$ takes the place of the collection of all possible balls $B(x, \epsilon)$, as x ranges over X and ϵ ranges over $(0, \infty)$.

Proposition. *Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X satisfying the following conditions:*

$$\text{(i)} \bigcup_{B \in \mathcal{B}} B = X;$$

(ii) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$.

Then there is a unique topology $\mathcal{T}$ characterized by:

(iii) $U \in \mathcal{T}$ iff for each $x \in U$ there is a $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Proof. First, I need to show that the collection $\mathcal{B}$ satisfies (i) and (ii).

For (i), it is clear that the union of all $B(x, \epsilon)$ must be at least X, since each $B(x, \epsilon)$ contains the point x for all x in X. To show that the sets are equal, consider $x \in \bigcup B$. Then, x must reside in at least one, B_1. Clearly, if x lives in an open ball of X, it must also reside in X. So, the union of all B's much also be contained in X.

For (ii), take two open balls $B_1, B_2 \in \mathcal{B}$ s.t. $x \in B_1 \cap B_2$. Since finite intersections of open sets are open, $B_1 \cap B_2$ is open. But given any such intersection, a smaller neighborhood can always be constructed such that $B_3 \subset B_1 \cap B_2$. *show with triangle ineq.*

So, $\mathcal{B}$ satisfies (i) and (ii). Now, to prove the proposition.

($\Longrightarrow$) Let $U \in \mathcal{T}$, then $U \subseteq X$ can be represented as some collection of neighborhoods B_α; that is, $U = \bigcup_\alpha B_\alpha$, since the B_α are open in X. Now, let $x \in U$. Since U is open, I can always find $\epsilon > 0$ s.t. $x \in B(x, \epsilon) \subset U$.

($\Longleftarrow$) Let $x \in U$ and let there be a set $B \in \mathcal{B}$ s.t. $x \in B \subset U$. This is true for all $x \in U$. Therefore I can represent U as a collection of these neighborhoods, that is $U = \bigcup_\alpha B_\alpha$. Since the union of open sets is itself an open set, $U \in \mathcal{T}$.

$\therefore$ there is a topology $\mathcal{T}$ characterized by the above definition. Proving uniqueness will come from part (b) of this exercise; since $\mathcal{B}$ is a base by definition.

(b) Prove Proposition 3.4.13. (This is part of problem 3.4.12)

If two bases B_1 and B_2 are equivalent, then they generate the same topology.

Proof. Assume $\mathcal{B}_1, \mathcal{B}_2$ are equivalent; that is, (i) $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and (ii) $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Given the proof from part (a), we know that the bases generate topologies. Use (i) to show that $\mathcal{T}_1$, the topology generated by $\mathcal{B}_1$, is finer than $\mathcal{T}_2$, and (ii) to show that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

Assume (i). Given any $x \in (B_2 \in \mathcal{B}_2)$, I can find $(B_1 \in \mathcal{B}_1) \ni x$ s.t. $x \in B_1 \subseteq B_2$. Therefore $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Assume (ii). Given any $x \in (B_1 \in \mathcal{B}_1)$, I can find $(B_2 \in \mathcal{B}_2) \ni x$ s.t. $x \in B_2 \subseteq B_1$. Therefore $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

$\therefore \mathcal{T}_1 = \mathcal{T}_2$. So, two equivalent bases generate the same topology.

This proves the uniqueness part of part (a).

(c) Recall that a topological space *second countable* if it has a countable base. Show that $\mathbb{R}$ with the usual topology is second countable.

Solution. Let $(\mathbb{R}, d)$ be the reals with the standard metric, and define $\mathcal{O} = \{(p,q)|p, q \in \mathbb{Q}$ to be a family of open intervals. I claim that this collection will form a countable base for all of the open sets in $\mathbb{R}$. Given that the rationals are dense in the reals and countable, then any open interval $(a,b) \in \mathbb{R}$ can be contained within some union of open sets $\bigcup_{O \in \mathcal{O}} O$, since I can find rational numbers $p \leq a$ and $q \geq b$.

Take the union of all such open sets, this will contain $\mathbb{R}$, and for any $O_1, O_2 \in \mathcal{O}$ and $x \in O_1 \cap O_2$ there exists some $O_3 \in \mathcal{O}$ such that $x \in O_3 \subset O_1 \cap O_2$, since the rationals are dense in the reals, and the rationals themselves are a subset of the reals, so clearly the rationals will be dense in itself. Therefore $\mathcal{O}$ is a base for $\mathbb{R}$, and we also know that $\mathcal{O}$ is countable. *show more rigorously*
$\therefore \mathbb{R}$ equipped with the usual topology is second countable, that is, it has a countable base.

(d) If $(X, \mathcal{T})$ is a second countable topological space, show that the topology is first countable.

Solution. Let $\mathcal{B}$ be a countable base that generates $\mathcal{T}$ on X. Consider the collection of sets

$$\mathcal{N}(x \in X) = \{B \in \mathcal{B} | x \in B\}.$$

It follows that $\mathcal{N}(x)$ is countable since it is a subset of a countable set. If this defines a local base, then $(X, \mathcal{T})$ is first countable.

By definition, $x \in V$ for all $V \in \mathcal{N}(x)$.

Suppose $V_1, V_2 \in \mathcal{N}(x)$. Since $\mathcal{B}$ satisfies the properties of a base, there must be a $V_3 \in \mathcal{N}(x)$ such that $V_3 \subseteq V_1 \cap V_2$.

Suppose $V \in \mathcal{N}(x)$. For $y \in V$, there must be $U \in \mathcal{N}(y)$ such that $U \subseteq V$. This follows from the observation that $V \in \mathcal{N}(y)$ and $\mathcal{B}$ satisfies the properties of a base for $V = V \cap V$. So, $\mathcal{N}(x)$ is a local base.

$\therefore$ a second countable topological space is also first countable.

(e) Problem 3.5.39 of Prof. Flaschka's notes.

A metric space (M, d) is said to be *separable* if there is a countable subset $\{x_n\} \subset M$ with the following property: given $x \in M$ and $\epsilon > 0$, there is an x_n such that $d(x, x_n) < \epsilon$. Prove that a metric space is second countable if, and only if, it is separable.

Proof. Suppose the metric topology generated by d on M is second countable. Then there is a countable base $\mathcal{B}$ for this topology. Let $\{B_n\}$ be some enumeration of $\mathcal{B}$. Construct a sequence $\{x_n\}$ s.t. $x_n \in B_n$ for all n. Since $\mathcal{B}$ is equivalent to the standard base of ϵ-balls, there must be $B_n \in \mathcal{B}$ s.t. $B_n \subseteq B(x, \epsilon)$ for any $x \in M$ and $\epsilon > 0$. Since $x_n \in B_n$, it follows that $d(x, x_n) < \epsilon$ and so (M, d) is separable.

Conversely, suppose (M, d) is separable. Consider the countable collection of sets

$$\mathcal{B} = \{B(x_n, 1/m) | m \in \mathbb{N}\},$$

where $\{x_n\}$ is the countable subset of M guaranteed by the definition. Each element of $\mathcal{B}$ is itself a metric ϵ-ball. Conversely, consider $B(x, \epsilon)$ for some $x \in M$ and $\epsilon > 0$. Then there is x_n s.t. $d(x, x_n) < \min(\epsilon/2, 1)$. As in Part (c), there will be $m \in \mathbb{N}$ s.t. $B(x_n, 1/m) \subseteq B(x, \epsilon)$. This implies $\mathcal{B}$ is equivalent to the normal base of ϵ-balls, so it generates the metric topology on M. So, the metric space (M, d) is second countable. $\therefore$ a metric space is second countable if and only if it is separable.

(f) Show that l^∞ with the metric topology is first countable, but not second countable.

Solution. To show that l^∞ is first countable, we merely need to refer to the first problem in this set, which states that every metric space, with its metric topology is first countable. The challenge here, is to show that it is not second countable. To do so, use previous result, and show that if it is not separable, then it also cannot be second countable.

First, consider the set $A \subseteq X$ of all sequences x_n such that each element of said sequence x_i consists of 0 or 1. We already know that this sequence is countable, using some manner constructing a sequence b that differs in each x_k in the k^{th} element (i.e. if the k^{th} element of x_k is a 0, then the corresponding entry of b is a 1). Thus b differs from every other x_k, yet b is always an element of A, thus A is uncountable. Lets see if this helps me show that l^∞ is not separable.

Suppose that $x, y \in A$, as defined above such that $x \neq y$. Thus $\|x - y\|_\infty = \sup_i |x_i - y_i| = 1$. Now, suppose that I can find some countable, dense subset U of X, thus U is also a countable dense subset of A. If I construct open balls in l^∞ such that each ball has radius $\frac{1}{8}$ and is centered at each point in U, which I will denote u_0. Given that these open balls are countable, while A is uncountable, I should be able to find some ball which contains the distinct elements $x, y \in A$. Thus

$$\begin{aligned} 1 &= \|x - y\|_\infty \\ &\leq \|x - u_0\|_\infty + \|u_0 - y\|_\infty, \text{ by the triangle inequality} \\ &< \frac{1}{8} + \frac{1}{8} \\ 1 &< \frac{1}{4} \end{aligned}$$

which is a contradiction. Thus, X equipped with l^∞ cannot be separable, and thus is not second countable.

2. Problem 3.4.14 of Prof. Flaschka's notes.

   An *open half-plane* in $\mathbb{R}^2$ is a set of the form

$$P(\mathbf{a}, r) \equiv \left\{(x_1, x_2)\middle| a_1x_1 + a_2x_2 > r\right\},$$

   where $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}$. Let $\mathcal{B}^0$ be the collection of all open half-planes. The collection $\mathcal{B}$ of all finite intersections of members of $\mathcal{B}^0$ is a base for a topology by Proposition 3.4.10. Show that this topology is just the usual metric topology on $\mathbb{R}^2$. (Hint: begin by drawing intersections of several sets from $\mathcal{B}^0$, say of the half-planes $x_1 < 1, x_1 > -1, x_2 < 1, x_2 > -2.$).

   Solution. From Problem 8.1 (b), it is sufficient to show that $\mathcal{B}$ is equivalent to the usual base of ϵ-balls for the metric topology on $\mathbb{R}^2$.

   Consider $B(\mathbf{x}, \epsilon)$ for some $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\epsilon > 0$. Construct $B \in \mathcal{B}$ by

$$\begin{aligned} B &= P((1,0), x_1 - \epsilon/4) \cap P((-1,0), -x_1 - \epsilon/4) \\ &\cap P((0,1), x_2 - \epsilon/4) \cap P((0,-1), -x_2 - \epsilon/4). \end{aligned}$$

   Then B is a square with side length $\epsilon/2$ centered at $\mathbf{x}$, so $B \subseteq B(\mathbf{x}, \epsilon)$.

   Conversely, consider $B \in \mathcal{B}$. For any $\mathbf{x} \in P(\mathbf{a}, r)$ with $\mathbf{a} \in \mathbb{R}^2$ and $r \in \mathbb{R}$, there must be some $\epsilon > 0$ such that $B(\mathbf{x}, \epsilon) \subseteq P(\mathbf{a}, r)$. So, $P(\mathbf{a}, r)$ is open in the metric topology on $\mathbb{R}^2$. This implies B is open, since it is a finite intersection of open half-planes. Thus, there is $\epsilon > 0$ s.t. $B(\mathbf{x}, \epsilon) \subseteq B$ for any $\mathbf{x} \in B$.

   $\therefore$ this topology is just the usual metric topology on $\mathbb{R}^2$.

3. Problem 3.4.31 of Prof Flaschka's notes.

   Let $\mathcal{B}^0$ be the collection of all subsets of $\mathbb{R}$ of the form $\mathbb{R} - \{x\}$, for some $x \in \mathbb{R}$. Show that $\mathcal{B}^0$ is a subbase for a topology on $\mathbb{R}$. Describe this topology as explicitly as possible.

   Solution. Let $x \in \mathbb{R}$. Then $x \in \mathbb{R} - \{y\}$ for any $\mathbb{R} \ni y \neq x$, so $x \in \bigcup_{B \in \mathcal{B}^0} B$.

   Let $x \in \bigcup_{B \in \mathcal{B}^0}$ B. Then $x \in\in \mathbb{R} - \{y\}$ for some $\{y\} \in \mathbb{R}$, so $x \in \mathbb{R}$.

   $\therefore \mathcal{B}^0$ is a subbase for a topology on $\mathbb{R}$ by Proposition 3.4.10.

   Elements of the base $\mathcal{B}$ associated with $\mathcal{B}^0$, finite intersection of sets from $\mathcal{B}^0$, are of the form $\mathbb{R} \setminus \{x_1, x_2, \ldots, x_n\}$ for $x_1, x_2, \ldots, x_n \in \mathbb{R}$. The topology generated by $\mathcal{B}$ is composed of arbitrary unions of such sets. The complement of this union is clearly contained in the complement of any of the sets in the union, and thus, unless the union is empty, it follows that the complement is finite. So, the topology generated by $\mathcal{B}$ is $\mathcal{T}_{\text{co-finite}}$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semi-continuous* if for all $x \in \mathbb{R}, \epsilon > 0$, there exists a $\delta > 0$ s.t. $|y - x| < \delta$ implies that $f(y) > f(x) - \epsilon$.

(a) Show that the collection $\mathcal{B} = \{(\alpha, \infty) | \alpha \in \mathbb{R}\}$ is a base. Let $\mathcal{T}'$ denote the topology generated by $\mathcal{B}$. Show that $\mathcal{T}' \subset \mathcal{T}_{\text{metric}}$ and the containment is strict. (Hint: One idea is to show that $\mathcal{T}'$ is not Hausdorff.)

Solution. Clearly, $\bigcup_{B \in \mathcal{B}} B = \mathbb{R}$.

Suppose $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. There are $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $B_1 = (\alpha_1, \infty)$ and $B_2 = (\alpha_2, \infty)$. If $B_3 = (\max(\alpha_1, \alpha_2), \infty)$ then $B_3 \in \mathcal{B}$ and $x \in B_3 \subseteq B_1 \cap B_2$.

$\therefore \mathcal{B}$ is a base.

Suppose $U \in \mathcal{T}'$. Then for all $x \in U$, there is $B_x = (\alpha_x, \infty) \in \mathcal{B}$ s.t. $B_x \subseteq U$. But then $B(\alpha_x + 1, 1/2) \subseteq B)_x \subseteq U$, so $U \in \mathcal{T}_{\text{metric}}$.

Consider $U = (0, 1) \in \mathcal{T}_{\text{metric}}$. For all $x \in U$, there is no $B_x = (\alpha_x, \infty) \in \mathcal{B}$ such that $B_x \subseteq U$. So $U \notin \mathcal{T}'$..

$\therefore \mathcal{T}' \subset \mathcal{T}_{\text{metric}}$ and the containment is strict.

(b) Show that the collection $\mathcal{B}$ along with the empty set and all of $\mathbb{R}$ is the topology generated by the base $\mathcal{B}$, *i.e.* $\mathcal{T}' = \mathcal{B} \cup \{\emptyset, \mathbb{R}\}$.

Solution. Since $\mathcal{T}'$ is a topology, $\emptyset$ and $\mathbb{R}$ are contained in $\mathcal{T}'$. Now, I need to show that every $U \in \mathcal{B}$ is also in $\mathcal{T}'$. From the proposition proven earlier, $U \in \mathcal{T}'$ iff $\forall x \in U, \exists B \in \mathcal{B}$ s.t. $x \in B \subset U$. Given any $U = (\alpha, \infty)$ which contains arbitrary x, I can always find an $\alpha' = \alpha + \epsilon$, $\epsilon > 0$, s.t. the set $B = (\alpha', \infty)$ will also contain x with the relationship $x \in B \subset U$. Therefore, $U \in \mathcal{T}'$. But, this relationship will hold for all sets U, since the choice of α is arbitrary, which implies that all elements of $\mathcal{B}$ are in $\mathcal{T}'$.

$\therefore \mathcal{T}' = \mathcal{B} \cup [\emptyset, \mathbb{R}]$.

(c) Show that $\mathcal{T}'$ is second countable.

Solution. To show that $\mathcal{T}'$ is second countable, I just need to find a countable base. For this, I pick the set

$$\mathcal{B}_{\mathbb{Q}} = \{(q, \infty) | q \in \mathbb{Q}\}.$$

Since I have already shown that $\mathcal{B}$ is a base, and $\mathbb{Q} \subset \mathbb{R}$, the second property of a base will hold. Now, I need to address

$$\bigcup_{B \in \mathcal{B}_{\mathbb{Q}}} B = \mathbb{R}?$$

Since the rationals are dense in $\mathbb{R}$, the equality follows directly from the argument made in part (a). Therefore, since $\mathbb{Q}$ is countable, $\mathcal{B}_{\mathbb{Q}}$ is a countable base.

$\therefore \mathcal{T}'$ is second countable.

(d) Show that a function $f : (\mathbb{R}, \mathcal{T}_{\text{metric}}) \to (\mathbb{R}, \mathcal{T}')$ is continuous, if and only if it is lower semi-continuous by the earlier definition.

Solution. Suppose f is continuous. Then for all $V \in \mathcal{T}'$ there is $U \in \mathcal{T}_{\text{metric}}$ s.t. $f(U) \subseteq V$. In other words, for all $x \in \mathbb{R}$ and $\epsilon > 0$, since $f(x) \in (f(x) - \epsilon, \infty) \in \mathcal{T}'$, there is $\delta > 0$ s.t. $|x - y| < \delta$ implies $f(y) > f(x) - \epsilon$. So, f is lower semi-continuous.

Conversely, if f is lower semi-continuous, if $f(x) \in (\alpha, \infty)$, it follows $\exists \epsilon > 0$ s.t. $\alpha < f(x) - \epsilon$, so $\exists \delta > 0$ s.t. $|y - x| < \delta \implies f(y) > \alpha$. This implies, if $x \in f^{-1}((\alpha, \infty))$, it is an interior point, so that the inverse image of any open set in $\mathcal{T}'$ is open in $\mathcal{T}_{\text{metric}}$. This proves f is continuous.

$\therefore$ f is continuous iff it is lower semi-continuous.

(e) Show that a function $f : (\mathbb{R}, \mathcal{T}') \to (\mathbb{R}, \mathcal{T}_{\text{metric}})$ is continuous, if and only if it is a constant function.

Solution. If f is a constant fucntion, then it must be continuous since the inverse image of any set is either $\emptyset$ or $\mathbb{R}$, so a constant function is continuous, no matter what topology we impose on the domain.

Suppose f is continuous. Then for all $V \in \mathcal{T}_{\text{metric}}$ there is $U \in \mathcal{T}'$ such that $f(U) \subseteq V$. In other words, for all $x \in \mathbb{R}$ and $\epsilon > 0$ there is $\delta > 0$ s.t. $y > x - \delta$ implies $|f(x) - f(y)| < \epsilon$. If f is not a constant function, then there are $x, y \in \mathbb{R}$ with $x < y$ s.t. $f(x) \neq f(y)$ This implies for some $\epsilon > 0$, $y > x - \delta$ but $f(x) - f(y)| > \epsilon$ for all $\delta > 0$. This is a contradiction, since f was assumed to be continuous. So, f is a constant function.

$\therefore$ f is continuous iff it is a constant function.